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Local Energy Flux and the Refined Similarity Hypothesis

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In this paper we demonstrate the locality of energy transport for incompressible Euler equations both in space and in scale. The key to the proof is the proper definition of a "local subscale flux," $\Pi_l(\mathbf{r})$, which is supposed to be a measure of energy transfer to length scales $<l$ at the space point $\mathbf{r}$. Kraichnan suggested that for such a quantity the "refined similarity hypothesis" will hold, which Kolmogorov originally assumed to hold instead for volume-averaged dissipation. We derive a local energy-balance relation for the large-scale motions which yields a natural definition of such a subscale flux. For this definition a precise form of the "refined similarity hypothesis" is rigorously proved as a big-O bound. The established estimate is $\Pi_l(\mathbf{r}) = O(l^{3h-1})$ in terms of the local Hölder exponent h at the point $\mathbf{r}$, which is also the estimate assumed in the Parisi–Frisch "multifractal model." Our method not only establishes locality of energy transfer, but it also clarifies the physical reason that convection effects, which naively violate locality, do not contribute to the subscale flux. In fact, we show that, as a consequence of incompressibility, such effects enter into the local energy balance only as the divergence of a spatial current. Therefore, they describe motion of energy in space and cancel in the integration over volume. We also discuss theorems of Onsager, Eyink, and Constantin *et al.* on energy conservation for Euler dynamics, particularly to explain their relation with the Parisi–Frisch model. The Constantin *et al.* proof may be interpreted as giving a bound on the total flux, $\Pi_l = \int d^d\mathbf{r}\, \Pi_l(\mathbf{r})$, of the form $\Pi_l = O(l^{z_3-1})$, where z_3 is the third-order scaling exponent (or Besov index), in agreement with the "multifractal model." Finally, we discuss how the local estimates are related to the results of Caffarelli–Kohn–Nirenberg on partial regularity for solutions of Navier–Stokes equations. They provide some heuristic support to a scenario proposed recently by Pumir and Siggia for singularities in the solutions of Navier–Stokes with small enough viscosity.

KEY WORDS: Turbulence; Navier–Stokes equations; multifractal model.

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1. INTRODUCTION

This paper is the third in a series of works which studies the inertial transfer of energy in incompressible fluids, relevant to high-Reynolds-number turbulent flow. In the first paper,[1] subtitled "Fourier analysis and local energy transfer" [hereafter referred to as (I)], we established the fundamental property of locality in scale of instantaneous energy transfer. Our proof was inspired by the Parisi–Frisch "multifractal model" of turbulence,[2,3] which postulates that individual realizations of the turbulent velocity field are Hölder continuous in the zero-viscosity limit, with Hölder exponent h occurring on a fractal set $S(h)$. We recall that the velocity field is said to have Hölder exponent h at space point $\mathbf{r}$, or $\mathbf{v} \in C^h(\mathbf{r})$, if

$$|\mathbf{v}(\mathbf{r}+\mathbf{l})-\mathbf{v}(\mathbf{r})| = O(l^h) \tag{1}$$

In fact, under a Hölder condition with exponent $0<h<1$, we established in (I) an estimate

$$\bar{\Pi}_\Lambda = O(2^{\Lambda(1-3h)}) \tag{2}$$

for an energy flux $\bar{\Pi}_\Lambda$ to wavenumbers $>2^\Lambda$. This estimate is local in scale, i.e., determined by the contribution of "local" wavevector triads. However, a surprise of our paper[1] was that the conventional measure of energy flux in k space, $\Pi(k)$, need not be dominated by the local triadic interactions. In fact, it was shown by an explicit example that very nonlocal triads can dominate in this conventional quantity through a process in which energy transfer between two large wavenumbers is "catalyzed" by a very low-wavenumber mode in the energy range. What we found in ref. 1 was that, if a "band-averaged flux" is defined as

$$\bar{\Pi}_\Lambda = \frac{1}{2^\Lambda}\int_{2^\Lambda}^{2^{\Lambda+1}} dk\, \Pi(k) \tag{3}$$

cancellations occur in the contributions from the nonlocal triads which are then appropriately weighted. It is therefore this averaged flux which is locally determined rather than the conventional one.

Although the argument in ref. 1 established local transfer, it nevertheless was based on a quite different physical mechanism than is often stated. A common belief is that the nonlocal triads describing convection of small-scale eddies by larger ones must be removed by going to "co-moving" or "Lagrangian" coordinates with subtract the purely convective contributions.[4],[2] Siggia[5] and Zimin[6] attempted to establish locality by making a theoretical analysis of the Navier–Stokes equations employing "comoving eddy" representations to attempt to remove the convective contributions of nonlocal triads. In the second paper of our series,[7] subtitled "Space-scale locality and semi-Lagrangian wavelets" [hereafter referred to as (II)], we examined this argument by constructing an exact version of such comoving eddies using a continuous wavelet representation of the Navier–Stokes dynamics. A main result of (II) was that the representation by "comoving eddies" does *not* suffice to remove the purely convective contributions from the energy transfer. We constructed a specific example for this representation, similar to that in ref. 1, in which the transfer is "catalyzed" by the large scales and is proportional to the amplitude of the large-scale velocity rather than its gradient. We furthermore showed that the previous heuristic arguments overlooked a fundamental difficulty of the "comoving eddy" representation, which is that "detailed energy conservation" does not hold for triads of comoving eddies.

Another problem in turbulence where wavelets would appear to be useful *a priori* is in the definition of "local subscale energy flux," i.e., a measure of energy transport to small length scales locally at a point in space. This is a fundamental issue of turbulence theory connected with the Kolmogorov "refined similarity hypothesis," which was originally proposed by Kolmogorov[8] as a relation

$$\varepsilon_l(\mathbf{r}) \sim \frac{[\Delta_l v(\mathbf{r})]^3}{l} \tag{4}$$

between the magnitude of the velocity difference $\Delta_l v(\mathbf{r})$ over length scale l at point $\mathbf{r}$ and the volume average $\varepsilon_l(\mathbf{r})$ of the local dissipation in a sphere of radius l centered at point $\mathbf{r}$. However, it was argued by Kraichnan in Section 6 of ref. 9 that a relation like Eq. (4) should hold for appropriately defined local fluxes $\Pi_l(\mathbf{r})$ rather than for space-averaged dissipation $\varepsilon_l(\mathbf{r})$. Recently, Meneveau[10] employed wavelets in an attempt to define a local flux $\Pi_a(\mathbf{r})$, measuring energy transport to eddies of size $<a$ at space point $\mathbf{r}$. However, we showed in (II) that there is a problem with Meneveau's

[2] The idea is often attributed to Kraichnan that locality of energy transfer is due to a possibility of removing convective interactions by transforming to a Lagrangian frame. However, this seems to be a misinterpretation. Kraichnan's point was instead that *time scales* in a Lagrangian description are intrinsic and locally determined (whereas Eulerian time scales are dominated by distant convective sweeping). He explicitly pointed out in his earliest work on DIA that locality of instantaneous transfer is obtained in the Eulerian representation as long as suitable scaling exists.

definition of the "local subgrid fluxes" $\Pi_a(\mathbf{r})$ because he included contributions from eddy triads with all three members of supergrid scale $>a$. A corrected definition was shown to be locally determined in space by the Hölder exponent at the given point, as required by the "multifractal model," but even the corrected definition failed to be locally determined in scale. The essential problem with the wavelet definitions of flux is that they fail to resolve wavenumbers smaller than the intrinsic spectral width of the wavelet and are therefore infected with spurious contributions from the large-scale modes.

Our papers (I) and (II) left unsolved the problem of defining an appropriate local subscale flux which would obey an estimate simultaneously local in space and scale. Therefore, the uncomfortable possibility remained that "local interactions" might dominate in the energy transfer only for a global quantity, averaged over space, and that locally in space the contribution of nonlocal triads might be the largest contribution. If that were so, then there would be little reason to believe in Kolmogorov's hypothesis of *universality of small-scale statistics*. Validity of that hypothesis would certainly require that the energy cascade be local in each small region of the flow and not merely in a spatially average sense. In this paper we derive an energy balance equation for the "large-scale motions" and we show that it yields immediately a natural prescription for a local flux $\Pi_l(\mathbf{r})$. The definition we use for the "large-scale" component of the velocity field, corresponding to length scales $>l$, is

$$\mathbf{v}_l(\mathbf{r}) = (\mathbf{v} * \varphi_l)(\mathbf{r}) \tag{5}$$

which is a convolution of $\mathbf{v}$ with a smooth mollifier $\varphi_l(\mathbf{r}) = l^{-d}\varphi(\mathbf{r}/l)$. This type of smooth filtering is precisely the method used in large-eddy simulation (LES) of turbulent flow (e.g., see refs. 11 and 12). It has also been used technically in a recent mathematical proof on energy conservation by Constantin *et al.* (CET),[13] which has partly motivated our work. It turns out to be convenient in our estimations to choose φ to have compact support in space, but there is actually a fairly great latitude in the choice of φ. For any of these choices we show that the flux $\Pi_l(\mathbf{r})$ so-defined obeys the correct space-scale locality estimates. Furthermore, the appearance of the flux in a local energy-balance relation confirms its physical interpretation as a "subscale flux" of energy to modes at distances $<l$. The argument greatly clarifies the role of convection processes in the energy balance, since these are shown to contribute only to the *spatial* transport of energy and not to the "downward" transfer to smaller length scales.

Our plan in this paper is as follows: In Section 2 we derive the local energy-balance relation. The physical significance of the various terms in the balance relation are explicated and shown to separate clearly into contributions to a "spatial current" and to a "subscale flux." In Section 3 we discuss the relation of the so-defined local flux to Kraichnan's version of the refined similarity hypothesis and also to the Parisi–Frisch "multifractal model." A basic space-scale local estimate for the flux is established in terms of the local Hölder exponent of the velocity field. In Section 4 we briefly review the CET conservation theorem from our perspective. We show that their estimate for the global flux is also that predicted by the "multifractal model" heuristics. Finally in Section 5 we explain how the estimates on local fluxes relate to the possibility of Navier–Stokes singularities at small viscosity.

2. LOCAL ENERGY BALANCE IN SPACE AND SCALE

In this section we derive the local balance relation for kinetic energy in scales $>l$,

$$e_l(\mathbf{r}, t) = \tfrac{1}{2}v_l^2(\mathbf{r}, t) \tag{6}$$

very much in the spirit of the local conservation laws of nonequilibrium thermodynamics (e.g., see Chapter II.3 of ref. 14). For generality of the discussion, we shall here consider the velocity field to be a solution of the Navier–Stokes equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p + \nu_0\,\Delta\mathbf{v} \tag{7}$$

with molecular viscosity ν_0. These equations are assumed to be obeyed in the sense of distributions, i.e., in weak sense. It therefore follows that the velocity field $\mathbf{v}_l$ for scales $>l$ obeys

$$\partial_t \mathbf{v}_l + \nabla\cdot(\mathbf{v}\otimes\mathbf{v})_l = -\nabla p_l + \nu_0\,\Delta\mathbf{v}_l \tag{8}$$

where the subscript l denotes here the smooth filtering operation by convolution, $f_l \equiv f * \varphi_l$. This equation holds at each instant if the solution possesses some continuity in time; otherwise, it requires some additional averaging in time.

A key identity, which was noted by CET,[13] is

$$(\mathbf{v}\otimes\mathbf{v})_l = \mathbf{v}_l\otimes\mathbf{v}_l + \mathbf{R}_l(\mathbf{v},\mathbf{v}) - (\mathbf{v}-\mathbf{v}_l)\otimes(\mathbf{v}-\mathbf{v}_l) \tag{9}$$

with

$$\mathbf{R}_l(\mathbf{v},\mathbf{v})(\mathbf{r}) \equiv \int d^d\mathbf{h}\,\varphi_l(\mathbf{h})[\Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})\otimes\Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})] \tag{10}$$

and

$$\Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r}) \equiv \mathbf{v}(\mathbf{r}-\mathbf{h}) - \mathbf{v}(\mathbf{r}) \tag{11}$$

The proof is elementary. The important point to observe is that, except for the first term in Eq. (9), the velocity appears only through its difference $\Delta_l \mathbf{v}$ over length scales $\sim l$ or through its small-scale component $\mathbf{v}'_l = \mathbf{v} - \mathbf{v}_l$. It is convenient to define

$$\mathbf{T}_l(\mathbf{v}, \mathbf{v}) \equiv \mathbf{R}_l(\mathbf{v}, \mathbf{v}) - \mathbf{v}'_l \otimes \mathbf{v}'_l \tag{12}$$

This term essentially depends only upon the modes at length scales $< l$ and can be bounded by using the Hölder conditions. The first term in the right-hand side of Eq. (9) contains the absolute velocity $\mathbf{v}_l$ and cannot be given a good bound.

Using this identity, it is easy to derive the following *energy balance relation*:

$$\partial_t e_l(\mathbf{r}, t) + \nabla \cdot \mathbf{j}_l(\mathbf{r}, t) = -\Pi_l(\mathbf{r}, t) - \varepsilon_l(\mathbf{r}, t) \tag{13}$$

The various terms appearing are

$$\varepsilon_l \equiv \nu_0 (\nabla \mathbf{v}_l)^2 \tag{14}$$

and

$$\Pi_l \equiv -(\nabla \mathbf{v}_l) : \mathbf{T}_l \tag{15}$$

where $\mathbf{A} : \mathbf{B} = \sum_{ij} A_{ij} B_{ij}$ and $\mathbf{A}^2 = \mathbf{A} : \mathbf{A}$. Also,

$$\mathbf{j}_l \equiv (p_l + e_l)\, \mathbf{v}_l + \mathbf{T}_l \cdot \mathbf{v}_l - \nu_0 \nabla e_l \tag{16}$$

Incompressibility was used essentially in writing the $\mathbf{j}_l$ term as a total divergence.

Each of the terms in the microscopic balance relation (13) has a precise physical interpretation. The term $\varepsilon_l(\mathbf{r}, t)$ represents the local dissipation of energy in the length scales $> l$ through the action of molecular viscosity. The term $\mathbf{j}_l(\mathbf{r}, t)$ is obviously a spatial current of energy in the scales $> l$, with contributions from convective transport by the velocity at those scales, proportional to $\mathbf{v}_l$, and from the tensor $\mathbf{T}_l$. This latter clearly represents a stress tensor, or pressure, due to the small-scale components $< l$, i.e., a kind of "Reynolds stress." Therefore, the contribution $\mathbf{T}_l \cdot \mathbf{v}_l$ to the energy current represents a spatial diffusion of energy in the large scales $> l$ due to "eddy viscosity" from the small-scale components $< l$. On the other hand, the term $\Pi_l(\mathbf{r}) = -(\nabla \mathbf{v}_l)(\mathbf{r}) : \mathbf{T}_l(\mathbf{r})$ represents a *local energy flux* from the large scales to the small scales. It gives the "effective dissipation" of energy in the large scales $> l$ due to the action of the eddy stress of the small scales $< l$ on the gradients of the large-scale motion.

3. REFINED SIMILARITY HYPOTHESIS AND THE MULTIFRACTAL MODEL

There is a very close relation of the discussion in the previous section with that of Kraichnan in Section 2 of ref. 9, where he considered also the problem of defining a "local energy flux." However, his proposed definition, $\tilde{\Pi}''(\mathbf{x}, t)$ in his Eq. (2.6), differs in a few important respects from our definition in Eq. (15). First, Kraichnan used a sharp cutoff in wavenumber. We have already seen in (I) that this is dangerous, and allows a spurious contribution from the large scales to dominate, even in the global flux $\tilde{\Pi}''$ defined by a volume average. Second, the quantity $\tilde{\Pi}''(\mathbf{x}, t)$ proposed by Kraichnan is actually of mixed character, including spatial transport of energy in addition to proper transfer of energy to the small-scale motions. The importance of the decomposition of the filtered stress tensor

$$(\mathbf{v}\otimes\mathbf{v})_l = \mathbf{v}_l \otimes \mathbf{v}_l + \mathbf{T}_l(\mathbf{v}, \mathbf{v}) \tag{17}$$

is that it allows a clean separation of these effects.

However, the main argument in Section 6 of Kraichnan's paper is confirmed by our analysis. He proposed there that a "refined similarity" relation should hold of the form

$$\Pi_l(\mathbf{r}) \sim \frac{[\Delta_l v(\mathbf{r})]^3}{l} \tag{18}$$

in place of Kolmogorov's hypothesis, Eq. (4). The reader is referred to that work for Kraichnan's very beautiful analysis of the matter. What we will show here is that the relation (18) holds in the following precise sense: that

$$\Delta_l v(\mathbf{r}) = O(l^h) \Rightarrow \Pi_l(\mathbf{r}) = O(l^{3h-1}) \tag{19}$$

at each space point $\mathbf{r}$. In other words, Eq. (18) leads to a correct big-O estimate of the flux $\Pi_l(\mathbf{r})$ in terms of the local Hölder exponent at the point $\mathbf{r}$. This is also a fundamental tenet of the Parisi–Frisch "multifractal model,"[2,3] which is here rigorously confirmed as well.

In fact, to establish Eq. (19), we need just the following elementary estimates:

$$\mathbf{R}_l(\mathbf{v}, \mathbf{v})(\mathbf{r}) = O(l^{2h}) \tag{20}$$

and

$$\mathbf{v}'_l(\mathbf{r}) = O(l^h) \tag{21}$$

and

$$\nabla\mathbf{v}_l(\mathbf{r}) = O(l^{h-1}) \tag{22}$$

when $\mathbf{v} \in C^h(\mathbf{r})$, i.e., when the local Hölder exponent of the velocity at point $\mathbf{r}$ is h. These estimates are quite easy to derive with our smoothness and support assumptions on φ_l. For the convenience of readers, we give brief derivations in an appendix.

Using now these estimates in conjunction with the definition

$$\Pi_l(\mathbf{r}) = -(\nabla \mathbf{v}_l)(\mathbf{r}) : \mathbf{T}_l(\mathbf{r}) \tag{23}$$

we can see directly that $\Pi_l(\mathbf{r}) = O(l^{3h-1})$ whenever $\Delta \mathbf{v}_l(\mathbf{r}) = O(l^h)$, as was claimed. It was obviously crucial in obtaining this result that our definition of $\Pi_l(\mathbf{r})$ contains only contributions from the small-scale modes $<l$ or from the large-scale gradient $\nabla \mathbf{v}_l$. On the other hand, the term $\nabla \cdot \mathbf{j}_l(\mathbf{r})$ does not obey an estimate like Eq. (19). Since it represents a space flux of energy, with a large contribution from pure convection, there is no physical reason to expect that it should. Much confusion on the subject of "local transfer" has arisen from a failure to distinguish between the different physical effects of transport in space and in scale.

It is very natural to attempt to model the small-scale stresses $\mathbf{T}_l$ by a gradient law in terms of the large-scale velocity field with a phenomenological "eddy viscosity" coefficient ν_l, as

$$\mathbf{T}_l(\mathbf{r}) = -\nu_l(\mathbf{r}) \cdot [\nabla \mathbf{v}_l(\mathbf{r}) + \nabla \mathbf{v}_l(\mathbf{r})^{\mathrm{T}}] \tag{24}$$

Consistency with the previous estimates is obtained if

$$\nu_l(\mathbf{r}) = O(l^{1+h}) \tag{25}$$

at a point where the local Hölder exponent is h.

4. ENERGY CONSERVATION AND BESOV SPACES

In our paper (I) we studied a claim from a 1949 work of Onsager,[15],[3] who pointed out the possibility of energy dissipation without molecular

[3] It may be worth pointing out here that Onsager's observation on energy dissipation for Euler equations is probably historically the first example of a *conservation-law anomaly*, similar to the well-known axial anomaly in quantum electrodynamics. In fact, the filtering approach we have used may be regarded as a smooth version of the "point-splitting regularization" used in field theory. In that case, the nonvanishing of the flux Π_l in the local energy-balance equation (13) for $l \to 0$ is entirely analogous to the anomaly that appears as a source term in local conservation laws of quantum fields, although it appears here for a classical field theory. The interpretation of turbulent fluxes as anomalies was suggested in a recent work by Polyakov,[16] which our argument makes precise. Note that the first derivation of an anomaly in relativistic quantum field theory was in the 1951 paper of Schwinger,[17] two years after Onsager's remark.

viscosity in ideal hydrodynamics and who gave also a sharp condition for conservation of energy by Euler dynamics. In particular, we proved by Fourier analysis a version of his claim: namely, that energy is conserved for a solution of Euler equations with a "$_*$-Hölder continuous" velocity, $\mathbf{v} \in C^h_*$, whose Fourier coefficients $A_{\mathbf{k}\alpha}$ obey the condition

$$\sum_{\mathbf{k},\alpha} |\mathbf{k}|^h \, |A_{\mathbf{k}\alpha}| < +\infty \tag{26}$$

when $h > 1/3$. Under this condition, we established the bound (2), which implies that $\lim_{\Lambda \to +\infty} \bar{\Pi}_\Lambda = 0$ when $h > 1/3$ and yields the result. However, Onsager actually proposed that the result is true for a velocity field obeying the standard Hölder condition, $\mathbf{v} \in C^h$, i.e.,

$$|\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}')| = O(|\mathbf{r} - \mathbf{r}'|^h) \tag{27}$$

when $h > 1/3$, whereas Eq. (26) is sufficient, but not necessary, for Eq. (27). In a later paper[19] we conjectured energy conservation to hold under even a weaker assumption than Onsager's, namely, under the condition that

$$\left[\int d^d\mathbf{r} \, |\mathbf{v}(\mathbf{r} + \boldsymbol{l}) - \mathbf{v}(\mathbf{r})|^p\right]^{1/p} = O(l^s) \tag{28}$$

for $p \geqslant 3$ and $s > 1/3$. This is a Hölder condition in L^p-mean sense, or a so-called *Besov condition.*[18] Using a Littlewood–Paley criterion for Besov spaces, it is actually quite easy to prove by the methods of (I) that energy is conserved for $s > 1/2$ and $p \geqslant 3$, but we did not find a proof of the stronger result for $s > 1/3$.

Subsequent to our work, a paper appeared by Constantin *et al.* (CET)[13] which solved this problem. In fact, those authors proved not only the original Onsager claim on energy conservation, but also our conjecture with just the weaker Besov condition (28). The strategy of the proof of CET is actually quite similar to that of our proof in (I), in that it is based upon the analysis of an energy flux Π_l to length scales $<l$ (although those authors did not use the language of "energy flux"). The flux of CET is defined by using a smoother averaging than our definition from (I), Eq. (3) above. In fact, they studied the quantity

$$\Pi_l \equiv -\frac{d}{dt} \int d^d\mathbf{r} \, \frac{1}{2} \, |\mathbf{v}_l(\mathbf{r})|^2 \tag{29}$$

using the same type of filtered field $\mathbf{v}_l$ as we defined in Eq. (5). Note that this is just the global flux corresponding to the local flux which we defined in Section 2:

$$\Pi_l = \int d^d\mathbf{r}\, \Pi_l(\mathbf{r}) \tag{30}$$

The key difference between the work of CET and ours, aside from our more physical perspective, is that our analysis is local and theirs was global. With these definitions, CET established

$$\Pi_l = O(l^{3s-1}) \tag{31}$$

when $\mathbf{v}$ satisfies the Besov condition (28) for $p=3$, which proved their result.

From the estimates given in the previous section it already follows that

$$\Pi_l = O(l^{3h-1}) \tag{32}$$

when $\mathbf{v} \in C^h$. This proves immediately the claim of Onsager for conservation under the standard Hölder condition with $h > 1/3$. It is hard to imagine a simpler proof of the theorem. (Historically, it is tempting to imagine that this was Onsager's own argument, since it is based upon a standard calculation in nonequilibrium thermodynamics, a subject Onsager pioneered.) Our own proof from (I) in a wavenumber representation established conservation only under the stronger condition on Fourier coefficients in Eq. (26). On the other hand, our detailed estimates in wavenumber space provided a little more information than just the "local estimate" (2). Namely, we established decay bounds also on the contributions from the "nonlocal triads," which is important to assess the size of corrections to local transfer at finite Reynolds number. In fact, it was shown in our paper (I) that the part of the energy flux $\bar{\Pi}_\Lambda$ due to direct transfer from the wavenumbers $\leqslant 2^{\Lambda-\Delta}$ is a fraction $O(2^{-(1-h)\Delta})$ of the total flux. Likewise, the part of the energy flux directly into the wavenumbers $\geqslant 2^{\Lambda+\Delta}$ is a fraction $O(2^{-2h\Delta})$ of the total flux. Although the dominant mechanism of energy transfer in scale is therefore the "local interactions" under the condition $\mathbf{v} \in C^h_*$ with $0 < h < 1$, the corrections to local transfer show only a slow decay, algebraic in wavenumber.

In addition, CET proved[13] the stronger result that energy is conserved for a (weak) solution of Euler equations if $\mathbf{v} \in B_3^{s,\infty}$ for any $s > 1/3$. The maximal value of $s = s_p$ for which $\mathbf{v} \in B_p^{s,\infty}$ is essentially a type of "multifractal exponent" for space averages, related to the scaling exponent z_p in

$$\int d^d\mathbf{r}\, |\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})|^p \sim l^{z_p} \tag{33}$$

as $z_p = p \cdot s_p$. (See our discussion in ref. 19.) The CET result was obtained from the bound

$$\Pi_l = O(l^{z_3 - 1}) \tag{34}$$

It is interesting to observe that this is exactly the estimate suggested by the "multifractal model" heuristics. Indeed, since $\Pi_l(\mathbf{r}) \sim l^{3h-1}$ when the Hölder exponent at point $\mathbf{r}$ is h, it is suggested that

$$\begin{aligned} \Pi_l &\sim \int \rho(dh)\, l^{(3h-1)+(d-D(h))} \\ &\sim l^{z_3 - 1} \end{aligned} \tag{35}$$

when $l \to 0$. We have used here the well-known formula[2]

$$z_p = \inf_h [ph + (d - D(h))] \tag{36}$$

This is proved to be a valid formula for the (maximal) Besov index in ref. 19, at least when $D(h)$ is taken to be a certain "box-counting dimension."

The proof of CET is not based upon this heuristic argument. Instead, they simply estimate Π_l by using the Hölder inequality and the bounds on L^p-norms:

$$\|\mathbf{R}_l(\mathbf{v}, \mathbf{v})\|_{L^{3/2}} = O(l^{2s}) \tag{37}$$

and

$$\|\mathbf{v}_l'\|_{L^3} = O(l^s) \tag{38}$$

and

$$\|\nabla \mathbf{v}_l\|_{L^3} = O(l^{s-1}) \tag{39}$$

when $\mathbf{v} \in B_3^{s,\infty}$. Derivations are included in the appendix for the convenience of the reader.

The same circle of mathematical ideas can be used to relate "scaling exponents" of the local flux to those of the velocity field. This is suggested by the relation to the refined similarity hypothesis discussed in Section 3. In fact, the original Kolmogorov form of the hypothesis[8] was introduced in order to relate scaling exponents τ_p of the volume-averaged dissipation,

$$\langle \varepsilon_l^p \rangle \sim l^{\tau_p} \tag{40}$$

for $l \to 0$, to the exponents ζ_p of the inertial-range structure functions, as

$$\zeta_p = p/3 + \tau_{p/3} \tag{41}$$

We find here a precise analog for the flux in the form that

$$\int d^d\mathbf{r}\, |\Pi_l(\mathbf{r})|^p = O(l^{z_{3p} - p}) \tag{42}$$

in terms of the velocity exponent z_p introduced in Eq. (33). For the proof we refer again to the appendix. If a maximal exponent t_p is introduced as the upper bound of those for which

$$\int d^d\mathbf{r}\, |\Pi_l(\mathbf{r})|^p = O(l^{t_p}) \tag{43}$$

then these results imply that

$$z_p \leqslant p/3 + t_{p/3} \tag{44}$$

This is entirely analogous to Eq. (41) except that we can derive rigorously only an inequality (because we deal with big-O bounds rather than true asymptotic scaling).

5. A RELATION TO NAVIER–STOKES SINGULARITIES

There is one rather surprising consequence which is suggested by the "multifractal model" estimates on local fluxes $\Pi_l(\mathbf{r})$. It is probably still generally believed that Navier–Stokes equations have smooth solutions for all time starting from smooth initial data with any positive viscosity ν_0. Instead, it is suggested by the "multifractal model" that singularities may occur when the viscosity is sufficiently small. Although the argument is heuristic and probably hard to make rigorous, we will present it here to stimulate further thought on the issue.

The key idea is that there is a "spectrum" of viscous cutoffs in turbulence associated to the various Hölder exponents of the velocity field in the flow.[20,21] At a point where the Hölder exponent is h the "multifractal model" predicts that the cutoff will scale with viscosity as

$$\eta_h \sim l_0 \left(\frac{\nu_0}{\nu_{0c}}\right)^{1/(1+h)} \tag{45}$$

Here l_0 is a finite length and ν_{0c} is a finite reference viscosity included to make the formula dimensionally correct. The idea behind this formula is that the "energy cascade" at point $\mathbf{r}$ will proceed down to a length scale $l=\eta(\mathbf{r})$ where the local flux $\Pi_l(\mathbf{r})$ and the viscous dissipation $\varepsilon_l(\mathbf{r})$ are of the same order of magnitude. This is equivalent to the matching of "eddy viscosity" with molecular viscosity

$$\nu_l(\mathbf{r}) \sim \nu_0 \tag{46}$$

using the gradient representation (24) of the small-scale stress tensor. The formula (45) above then follows immediately by inverting Eq. (25) with $l=\eta_h$.

It is easy to see that ν_{0c} plays the role of a "critical viscosity": if $\nu_0 < \nu_{0c}$, then $\eta_h \to 0$ as $h \to -1$. This suggests that there may be singularities for the Navier–Stokes dynamics at finite viscosities $\nu_0 < \nu_{0c}$ if the Euler equations can generate an $h=-1$ singularity. Coincidentally, this is just the Hölder exponent of a singular vortex filament, which could be produced by vortex-stretching processes in the inertial dynamics. This speculation seems to be consistent with known facts on regularity of Navier–Stokes solutions. It was proved already by Leray[22] that the Navier–Stokes equations have global smooth solutions if the viscosity is large enough (see also Ladzhenskaya[23]). This is compatible with existence of a critical ν_{0c}. Also, the Caffarelli–Kohn–Nirenberg result[24] on partial regularity allows a singularity set in space-time of Hausdorff dimension 1 (but one-dimensional Hausdorff measure zero). Their argument furthermore establishes that any singularity which might appear must have a minimum rate of blowup $|\mathbf{v}| \geqslant C/r$ for distance from the singularity $r \to 0$ (see Section 5 of ref. 24 for a precise statement). The relation of this blowup rate to the $h=-1$ we obtained heuristically is not accidental, since both arguments are based upon a local energy balance. Therefore, all the rigorous results are consistent with "filamentlike" singularity sets occurring at a discrete set of times. Such a scenario for Navier–Stokes singularities is remarkably similar to one advanced by Pumir and Siggia[25] on the basis of a vortex filament simulation. In their picture the irregular points of the velocity will be a true space-time set, traced out by a "pinch singularity" which runs up a pair of antiparallel vortex lines. However, recent results of Constantin–Fefferman indicate this is an unlikely situation for singularity formation.[26]

APPENDIX. BASIC ESTIMATES

A standard mollifier φ is taken to be nonnegative with unit integral:

$$\int d^d\mathbf{r}\, \varphi(\mathbf{r}) = 1 \tag{A1}$$

It is chosen here also to be supported in the unit ball and to have some smoothness (one continuous derivative is actually sufficient). By definition

$$\mathbf{v}_l(\mathbf{r}) = \int d^d\mathbf{r}'\ \mathbf{v}(\mathbf{r}')\ \varphi_l(\mathbf{r}-\mathbf{r}') \tag{A2}$$

Therefore, with $\mathbf{v}_l' = \mathbf{v} - \mathbf{v}_l$,

$$\begin{aligned}\mathbf{v}_l'(\mathbf{r}) &= \int d^d\mathbf{r}'\ [\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}')]\ \varphi_l(\mathbf{r}-\mathbf{r}') \\ &= -\int d^d\mathbf{h}\ \Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})\ \varphi_l(\mathbf{h})\end{aligned} \tag{A3}$$

Using the condition $\mathbf{v} \in C^h(\mathbf{r})$, i.e.,

$$|\Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})| \leqslant (\text{const})\ |\mathbf{h}|^h \tag{A4}$$

it follows directly from Eq. (A3) that

$$\begin{aligned}|\mathbf{v}_l'(\mathbf{r})| &\leqslant (\text{const}) \int d^d\mathbf{h}\ \varphi_l(\mathbf{h})\ |\mathbf{h}|^h \\ &= O(l^h)\end{aligned} \tag{A5}$$

Since

$$\mathbf{R}_l(\mathbf{v}, \mathbf{v})(\mathbf{r}) = \int d^d\mathbf{h}\ \varphi_l(\mathbf{h})[\Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r}) \otimes \Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})] \tag{A6}$$

the same arguments give

$$|\mathbf{R}_l(\mathbf{r})| = O(l^{2h}) \tag{A7}$$

Finally, using $\int \nabla\phi = 0$,

$$\begin{aligned}\nabla\mathbf{v}_l(\mathbf{r}) &= \int d^d\mathbf{r}'\ (\nabla\varphi_l)(\mathbf{r}-\mathbf{r}')[\mathbf{v}(\mathbf{r}') - \mathbf{v}(\mathbf{r})] \\ &= \int d^d\mathbf{h}\ (\nabla\varphi_l)(\mathbf{h})\ \Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})\end{aligned} \tag{A8}$$

Applying the same estimates as before, we have

$$|\nabla\mathbf{v}_l(\mathbf{r})| = O(l^{h-1}) \tag{A9}$$

We note as an aside that the identity (A3) yields the following simple expression for the small-scale stress tensor

$$\mathbf{T}_l(\mathbf{r}) = \langle \Delta_{(\cdot)}\mathbf{v}(\mathbf{r}) \otimes \Delta_{(\cdot)}\mathbf{v}(\mathbf{r}) \rangle_l - \langle \Delta_{(\cdot)}\mathbf{v}(\mathbf{r}) \rangle_l \otimes \langle \Delta_{(\cdot)}\mathbf{v}(\mathbf{r}) \rangle_l \tag{A10}$$

Here $\langle \cdot \rangle_l = \int \varphi_l(\cdot)$ denotes an average over the separation distance with respect to the filter.

We now consider the estimates under the Besov condition $\mathbf{v} \in B_p^{s,\infty}$, which may be stated as

$$\|\Delta_{\mathbf{h}}\mathbf{v}\|_{L^p} = O(|\mathbf{h}|^s) \tag{A11}$$

From Eq. (A3), it follows by Jensen's inequality for $p > 1$ that

$$|\mathbf{v}_l'(\mathbf{r})|^p \leqslant \int d^d\mathbf{h}\ \varphi_l(\mathbf{h})\ |\Delta_{\mathbf{h}}\mathbf{v}(\mathbf{r})|^p \tag{A12}$$

Integrating over $\mathbf{r}$, it follows that

$$\|\mathbf{v}_l'\|_{L^p}^p \leqslant \int d^d\mathbf{h}\ \varphi_l(\mathbf{h})\ \|\Delta_{\mathbf{h}}\mathbf{v}\|_{L^p}^p \tag{A13}$$

from which the estimate

$$\|\mathbf{v}_l'\|_{L^p} = O(l^s) \tag{A14}$$

is immediately obtained. Using the Cauchy–Schwartz inequality and arguing in the same manner as above it follows from the expression (A6) that

$$\|\mathbf{R}_l\|_{L^{p/2}}^{p/2} \leqslant \int d^d\mathbf{h}\ \varphi_l(\mathbf{h})\ \|\Delta_{\mathbf{h}}\mathbf{v}\|_{L^p}^p \tag{A15}$$

This gives the estimate

$$\|\mathbf{R}_l\|_{L^{p/2}} = O(l^{2s}) \tag{A16}$$

The final estimate is only slightly more difficult. Using Eq. (A8), it can be shown that

$$\|\nabla\mathbf{v}_l\|_{L^p} = O(l^{s-1}) \tag{A17}$$

Jensen's inequality can still be applied if the mollifier φ is chosen to be spherically symmetric and monotonically decreasing in the radial coordinate (the integrals must be separated into two terms, corresponding to increasing and decreasing parts along a rectilinear coordinate direction). The details are left to the reader.

For the purpose of deriving these results with filter functions φ used in practical LES modeling, we remark that quite modest regularity actually suffices. For example, the local results hold if $\varphi(\mathbf{r})$ and $\nabla\varphi(\mathbf{r})$ are both continuous and have a finite $|\mathbf{r}|^{2h}$ moment. The commonly used Gaussian filter is a "good" example which easily satisfies these constraints. However, the sharp cutoff filter in Fourier space does not and, in fact, Example 1 in ref. 1 shows that the bounds may indeed be violated for this filter. These points are discussed at length in another work.[27]

Finally, let us derive the bound claimed at the end of Section 4 on moments of the local flux, Eq. (42). In fact, noting that $1/p = 1/(3p) + 2/(3p)$, we have that

$$\|\Pi_l\|_p \leqslant \|\nabla \mathbf{v}_l\|_{3p} \, \|\mathbf{T}_l\|_{3p/2} \tag{A18}$$

follows from the (generalized) Hölder inequality. Applying the previous estimates (A14), (A16), and (A17) then gives, for each $\varepsilon > 0$,

$$\|\Pi_l\|_p = O(l^{3s_{3p} - 1 - \varepsilon}) \tag{A19}$$

which is the precise statement of Eq. (42).

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